On the Lunar Apogee and the Black Moon

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Abstract

The lunar apogee is an astronomical factor used in certain areas of Astrology, where it is assumed to play a role in the reading of properties and situations indicated by the birth chart. In Astrology, the apogee is usually referred to as the Black Moon, or Lilith. In certain areas of Astrology, it is customary to use two black moons, the Black Moon as such and its 'real' or corrected version. Already for decades there has been uncertainty as to the precise position and motion in the zodiac of this corrected point, relative to its uncorrected counterpart. In this paper we start from the current situation and develop a simple but quite accurate approximation formula for the longitude correction of the corrected Black Moon, given by the ephemerides of Duval and Font. The obtained correction formula is subsequently used to derive formula's for the longitudinal velocity and acceleration of the corrected Black Moon.

1 Introduction

The Moon orbits Earth, and Earth has an orbit around the Sun. Besides Earth and its moon, there are several other planets, asteroids and dwarf planets, with their satellite moons circling around the Sun in our solar system. All these planets and their satellites exert gravitational forces onto one another, thereby influencing each other's orbits. In two-dimensional projections these orbits are nicely described by ellipses, closed curves with two focal points, as did Johannes Kepler state in his first law. Elliptical orbits of planets and satellites have one focal point occupied by the Sun or a planet and the other focal point empty. Without any disruption or interference, the position farthest from the occupied focal point will be on the straight line through the two focal points. If we restrict ourselves to the system of three bodies Sun, Earth, and Moon, it is clear that the Sun exerts an enormous gravitational force on the Moon in its orbit around Earth. However, other planets, notably the planet Jupiter, also exert gravitational forces on our moon. Therefore, it is not surprising that the Moon in its orbit usually does not pass the theoretical farthest point of the ellipse, but passes the apogee at some other position in space. Moreover, the Moon's orbit is three dimensional, i.e.

the Moon moves around Earth in an orbit that resembles an ellipsoidal spiral around Earth's orbit around the Sun. The Moon's orbit regarded as an ellipse, also exhibits a rotation of the ellipse around Earth. In short, a very complex motion, that is projected onto two dimensions in the zodiac.

In Astrology, the empty focal point of the lunar orbit is called Black Moon [1, p. 93 et seq.]. The name Lilith is also used, although often calculated and interpreted differently. The Astrology of the Invisible Luminaries has assigned an important role to the so called corrected Black Moon. Does this correction imply the determination of the exact position of the lunar apogee in the zodiac? This is not clear. Already for decades, the ephemerides of Duval and Font [2] –in the sequel simply referred to as the ephemerides– are considered the 'bible' in this respect. In the introduction of their book, Duval and Font state that in their calculations only the main term of all correction terms of the orbital equations has been taken into account. This main term would have an amplitude of $11° 60$ ($12°$?). Moreover do they justify the withholding from publication of their formulas with the argument of protection against plagiarism. Furthermore, and important in the context of this paper, they state that due to the varying eccentricity of the lunar orbit the Moon is not in its apogee when it is in conjunction with Lilith, the name they assign to the point on the ellipse farthest from earth. These three statements have been raising questions for decades, and today still do. What is the foundation of the longitude correction of the Black Moon? Are Duval and Font's ephemerides correct in this respect? And what is the relation between the true lunar apogee and the longitude correction of Duval and Font? Before starting to think about answering these questions, I began to devise a formula for the correction that would be in close agreement with the ephemerides. This is the topic of the next section. It turns out that de formula I propose is very accurate, in fact it is accurate within a few arcminutes from the ephemerides.

The reader who wonders why such an accuracy is relevant, be pointed to the role of the corrected Black Moon in progressive horoscopes, for which velocities of degrees per day in real time are taken as degrees per year. The corrected Black Moon exhibits a much faster motion than its uncorrected counterpart, moving around it, partly retrograde. An inaccuracy in position may result in an inaccuracy of several months in the progressive horoscope.

The next section describes a mathematical model for Duval and Font's longitude correction of the Black Moon. Section 3 provides the support of the mathematical model by application of spectral analysis. In Section 4 the motion of the corrected Black Moon in the zodiac is further analysed, resulting in accurate formulae for its velocity and acceleration. Finally, conclusions are given in Section 5. In the remainder of this paper, BM and cBM will denote Black Moon and corrected Black Moon respectively.

2 A Model of the correction by Duval and Font

In his book on Invisible Luminaries George Bode gives a model for calculating the correction of the Black Moon by means of a table [1, p. 239]. The table gives values

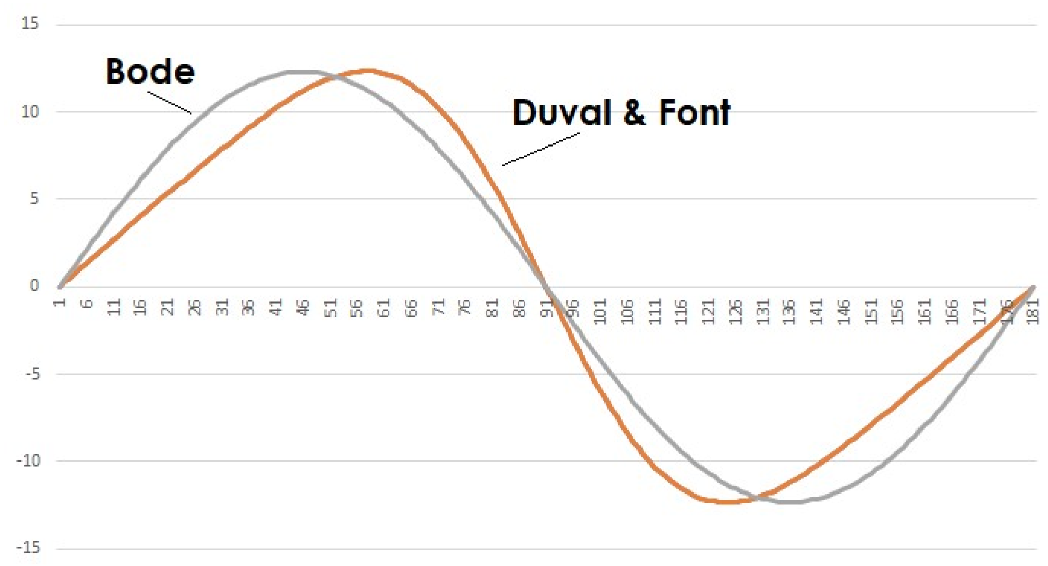

Figure 1: Longitude correction angle of Bode and Duval & Font.

for the correction in degrees as a function of the distance in the zodiac between the Sun and the BM. It turns out that this table implements a simple sinusoidal relation between this distance and the correction angle. Let φ denote the angle between the sun and the BM, and let φ_{cB} denote the correction angle in degrees according to Bode's table. The following equation then holds.

$$\varphi_{cB} = 12.3 \sin(2\varphi). \tag{1}$$

In the past, Bode already distanced himself from the table proposed in his book for reasons of inadequacy in the application to progressive horoscopes, due to the discrepancy with the ephemerides. A table of typical values of the correction angle has been extracted [3] from the ephemerides. Figure 1 gives a graphical representation of these correction angles, denoted by φ_{cD}, together with the values of Bode's sinusoidal relation.

The graph of φ_{cD} shows antisymmetry with respect to $\varphi = 90°$, similar to the sine function. Closer inspection of the graph of φ_{cD} shows a variable lag with the sine function, i.e. an increasing lag until the function reaches its maximum value, and a decreasing lag until the function value reaches zero at $\varphi = 90°$. Also, φ_{cD} reaches its maximum for $\varphi \approx 57°$. Two arguments have led me to devise a formula for φ_{cD}. First, the lag behaves like a sine function itself starting at zero, increasing to a maximum halfway and decreasing to zero again. Secondly, positions in the zodiac are projections of 3D space onto flat surfaces, which involve trigonometric functions. The analytical formula proposed is given by Equation (2)

$$\varphi_{cD} = 12.37 \sin\left[2(\varphi - 11.72 \sin(2\varphi))\right]. \tag{2}$$

The numbers 12.37 and 11.72 in (2) are obtained by manual optimization for the smallest maximal deviation from the given correction values of the ephemerides. It turns out that for these numbers, the maximal deviation amounts to 10.1 arcminutes

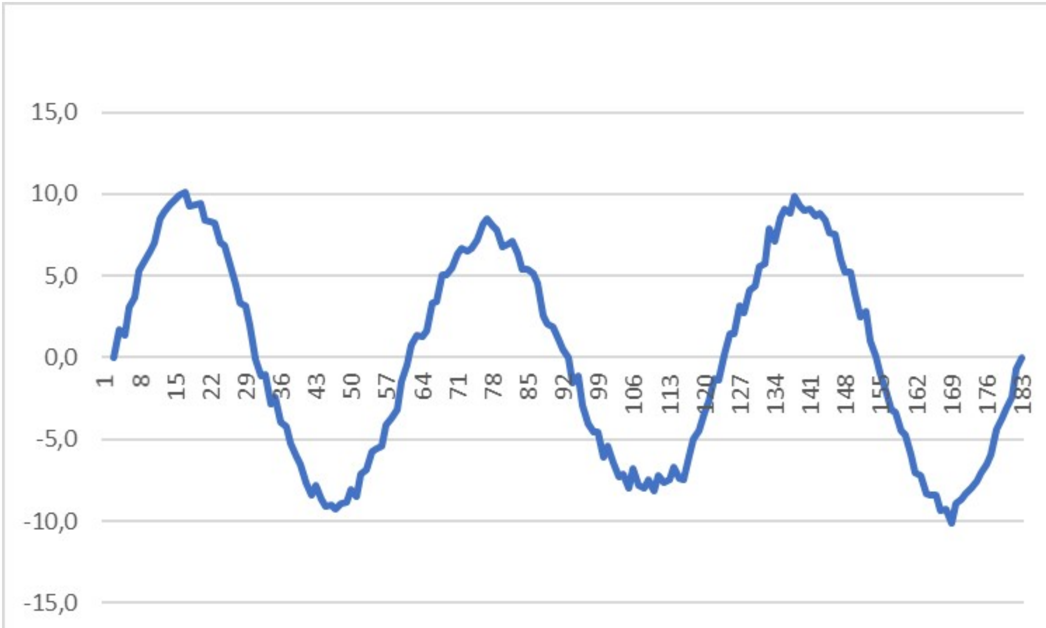

Figure 2: Deviation in arcminutes of φ_{cD} from ephemerides.

only. Further optimization using the mean square error criterion results in a small adjustment of the second number to 11.726. The deviation of φ_{cD} according to (2) with respect to the table of correction values from the ephemerides is shown graphically in Figure 2. The graph clearly shows the noise from the rounded off values in the ephemerides, but -surprisingly- also a remarkable third harmonic sine like function that seems present in the ephemeride values. What could be the source of this third harmonic disturbance? For now, this question remains unanswered. However, this third harmonic can be taken into account to adjust Equation 2 to obtain an even more accurate approximation formula.

2.1 Refinement of the model

The small deviation shown in Figure 2 is an invitation to refine the model described, by including a third harmonic to largely compensate for its presence in the data from the ephemerides. By simple addition of a term with the sine of 6φ the accuracy is increased, exhibiting a maximal deviation of only 2 arcminutes. The equation for φ_{cD} then becomes as shown by (3).

$$\varphi_{cD} = 12.37 \sin\left[2(\varphi - 11.72 \sin(2\varphi))\right] + (8.8/60)\sin(6\varphi)). \tag{3}$$

The amplitude of 8.8 arcminutes of the third harmonic is again the result of manual optimization. The resulting accuracy of 2 arcminutes is sufficient for most applications in astrology. Figure 3 shows the very precise approximation of the deviation mentioned before (the grey curve). The orange curve is the third harmonic sine, and the blue curve is the graph of the final remaining deviation. This blue curve clearly contains much rounding noise, but seems to also contain a fourth harmonic of very small amplitude.

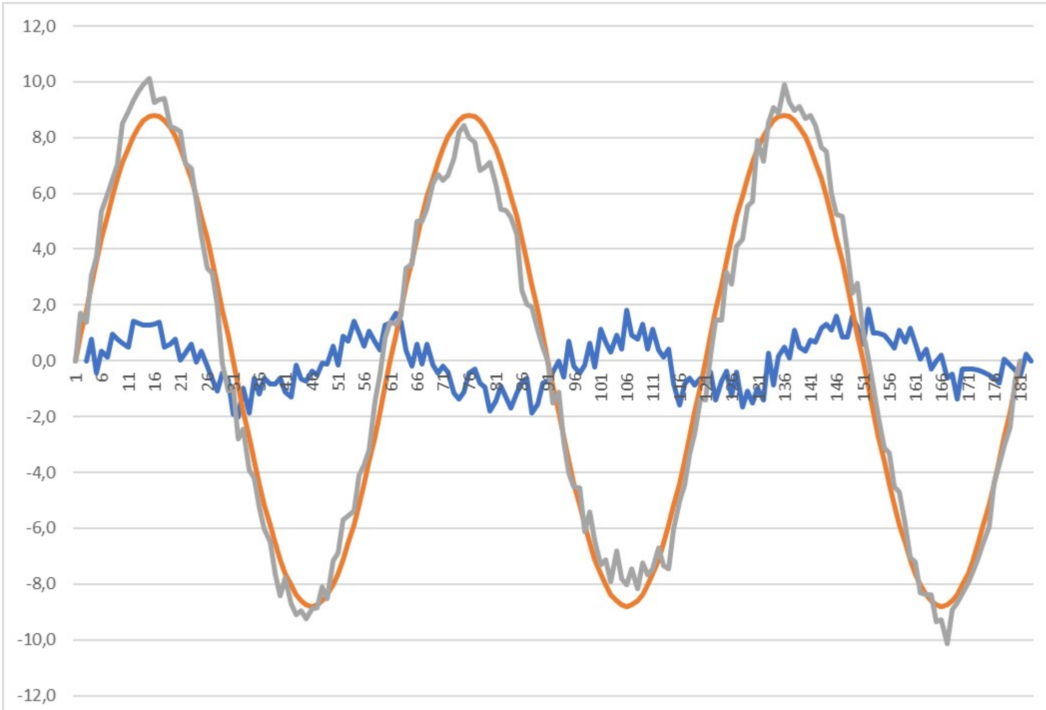

Figure 3: Approximation by third harmonic and remaining deviation.

3 Support by spectral analysis

A periodic function like $\varphi_{cD}(\varphi)$ can be transformed into a spectrum, a series of amplitudes of sines and cosines of harmonics that approximate the function. The spectrum also shows the contributions of the individual harmonics, or equivalently in more popular wording, the frequencies and their amplitudes present in the function. This transformation is achieved with the Discrete Fourier Transform [4] (DFT). Discrete here means that the function is given by a finite number of table entries. The antisymmetry of the correction values around 0° implies sine harmonics only, although small amplitude cosine terms may occur due to round offs. Spectral analysis of the data shows three dominant spectral components, viz. the ground 'frequency' 2φ and the second and third harmonics, both for the ephemeride values and φ_{cD} according to (2) and (3). The spectrum is depicted in Figure 4 which shows the almost perfect match of φ_{cD} from (3) with the table data. The conclusion of the spectral analysis must be that the proposed formula (3) is a very accurate approximation of the longitude correction angle according to the ephemerides. A further DFT analysis of the differences of Equations (2) and (3) with the table data is shown in Figure 5. This figure shows the fourth harmonic with very small amplitude, and higher harmonics with negligible amplitudes due to rounding off and inaccuracies in the calculations. Further attempts to remove the fourth harmonic did not result in greater accuracy.

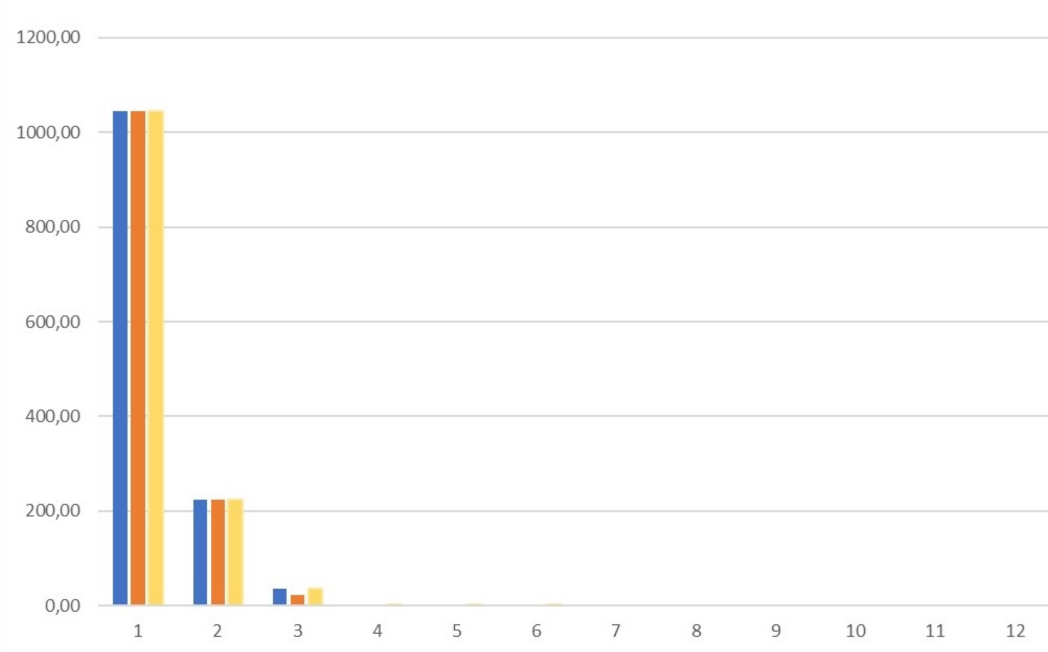

Figure 4: Spectra of correction from table data (blue), from Equation (2) (orange), from (3) (yellow)

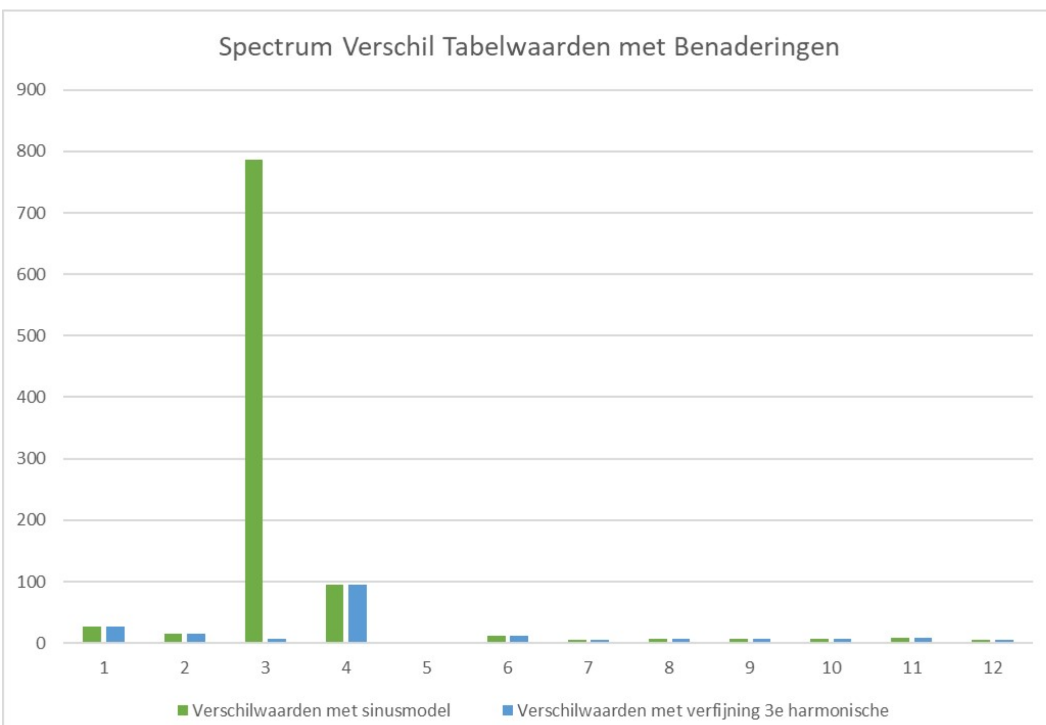

Figure 5: Spectra of differences between table data and φ_{cD} according to (2) (green) and (3(blue)

4 The motion of the Black Moon in the zodiac

In the previous section an analytical formula is given for the longitude correction angle according to the ephemerides of Duval and Font. It is shown to be sufficiently accurate for practical astrological purposes. In the sequel, φ_{cD} according to (3) will be taken for further analysis. This function will be denoted by $F(\varphi)$. The ephemerides do not list velocities, let alone accelerations of the cBM, a serious drawback. In this section equations are given for the velocity and acceleration of the cBM. Once a mathematical formula is available for the position of an object as a function of time, it is a relatively simple exercise to obtain formulas for the velocity and acceleration of the object. To this end, the first and second time derivatives of the position function have to be determined.

4.1 Velocity

Recall that the longitude correction angle φ_{cD} is a function of φ, the longitude difference of the Sun, φ_S, and the (uncorrected) Black Moon, φ_{BM}. The longitudes of Sun and BM vary with time, i.e. are functions of time. The absolute position of the cBM, φ_{cBM}, is the sum of the position of the BM and φ_{cD}, hence, the following equation holds.

$$\varphi_{cBM}(t) = \varphi_{BM}(t) + F(\varphi(t)), \text{with } \varphi(t) = \varphi_S(t) - \varphi_{BM}(t). \tag{4}$$

The equation for velocity of the cBM is obtained by taking the time derivative of (4).

$$\begin{aligned} v_{cBM}(t) = \frac{d}{dt}\varphi_{cBM}(t) &= \frac{d}{dt}\varphi_{BM}(t) + \frac{d}{d\varphi}F(\varphi)\frac{d}{dt}\left(\varphi_S(t) - \varphi_{BM}(t)\right) \\ &= v_{BM}(t) + \frac{d}{d\varphi}F(\varphi)\left(v_S(t) - v_{BM}(t)\right). \end{aligned} \tag{5}$$

Next, the velocities of the Sun and BM are taken constant, which implies that the Sun travels through the zodiac at $360/365.25 = 0.985626$ $(59^\circ\,8'\,25'')$ degrees per day. The BM completes a full round through the zodiac in 8 years, 10 months, and 5.4 days, which comes down to 0.111394 $(0^\circ\,6'\,41'')$ degrees per day. Looking at the relative motion of the corrected BM with respect to the uncorrected BM, the second term of Equation (5) suffices. In particular, the derivative of the correction angle $\frac{d}{d\varphi}F(\varphi)$, which will be denoted as $F'(\varphi)$.The derivation of F' is described in the appendix. The function $F'(\varphi)$ is dimensionless, its graph is shown in Figure 6. In this figure, the orange curve is the graph of $F'(\varphi)$, the derivative to φ of φ_{cD} of (3). The blue curve is extracted from the ephemerides, and clearly illustrates the effects of rounding off. For comparison with Bode's model the grey curve, a cosine function, is shown. The velocity of the cBM through the zodiac, relative to the uncorrected BM amounts to a maximum of 0.2704 times the difference of the velocities of the Sun and the Black Moon and a minimum of -0.62388 times this difference. The graph of the absolute velocity according to Equation (5) is shown in Figure 7. This graph shows a maximal velocity of 0.34780 degrees $(0^\circ\,20'\,52'')$ per day in direct motion, and -0.43402 degrees $(-0^\circ\,26'\,2.5'')$ per day in retrograde motion. The actual absolute velocity may be

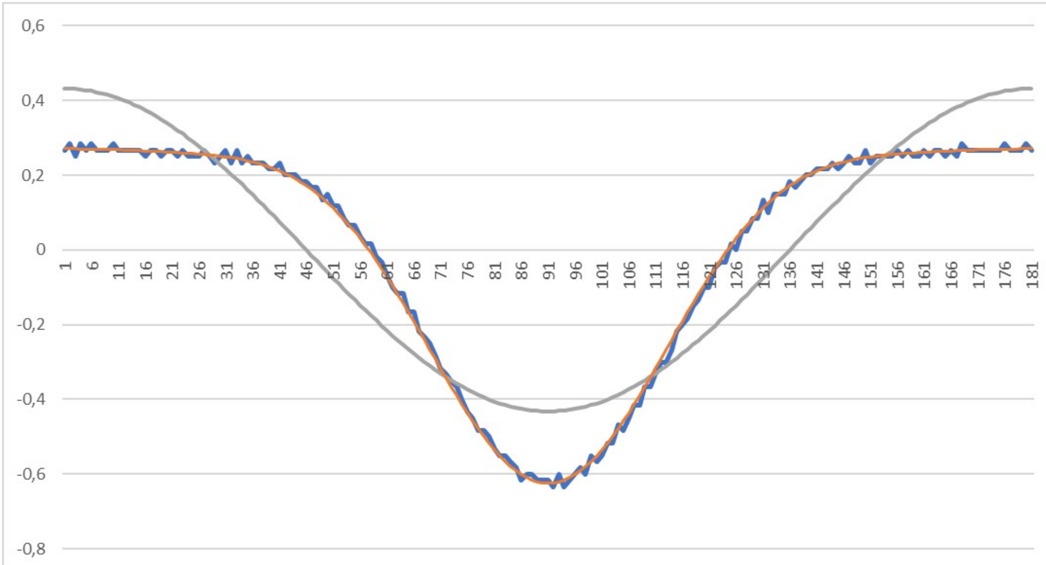

Figure 6: Graph of $F'(\varphi)$. Analytical (orange), delta's from ephemerides (blue), and cosine (grey)

slightly higher or lower, depending on the velocity of the Sun. Clearly visible in the graph is the short period of retrograde motion. The time it takes for the Sun and the Black Moon to move from conjunction to opposition is slightly less than seven months on average. In this period, the corrected Black Moon is in retrograde motion during slightly more than two months.

4.2 Acceleration

In a similar way as described in the previous subsection, i.e. by calculation of the time derivative of the velocity, the acceleration of the corrected Black Moon's motion through the zodiac is determined. By differentiating Equation 5 the acceleration is obtained as follows.

$$\begin{aligned} a_{cBM}(t) &= \frac{d}{dt} v_{cBM}(t) \\ &= \frac{d}{dt} \left(v_{BM}(t) + F'(\varphi) \left(v_S(t) - v_{BM}(t) \right) \right) \\ &= a_{BM}(t) + \frac{d}{d\varphi} F'(\varphi) \left(v_S(t) - v_{BM}(t) \right)^2 + F'(\varphi) \left(a_S(t) - a_{BM}(t) \right) \\ &= F''(\varphi) \left(v_S(t) - v_{BM}(t) \right)^2 \end{aligned} \tag{6}$$

In the above equation $F''(\varphi)$ is the second derivative of the longitude correction angle φ_{cD}, and the accelerations of Sun and Black Moon are zero, as their velocities are taken constant. The acceleration is shown in the graph of Figure 8, and described in the appendix. The maximal amplitude of the acceleration –and deceleration– is 0.0193 degrees ($0° \, 1' \, 9.4''$) per day-square. This very small acceleration is less than the accuracy of the data listed in the ephemerides, that are clearly totally inadequate to

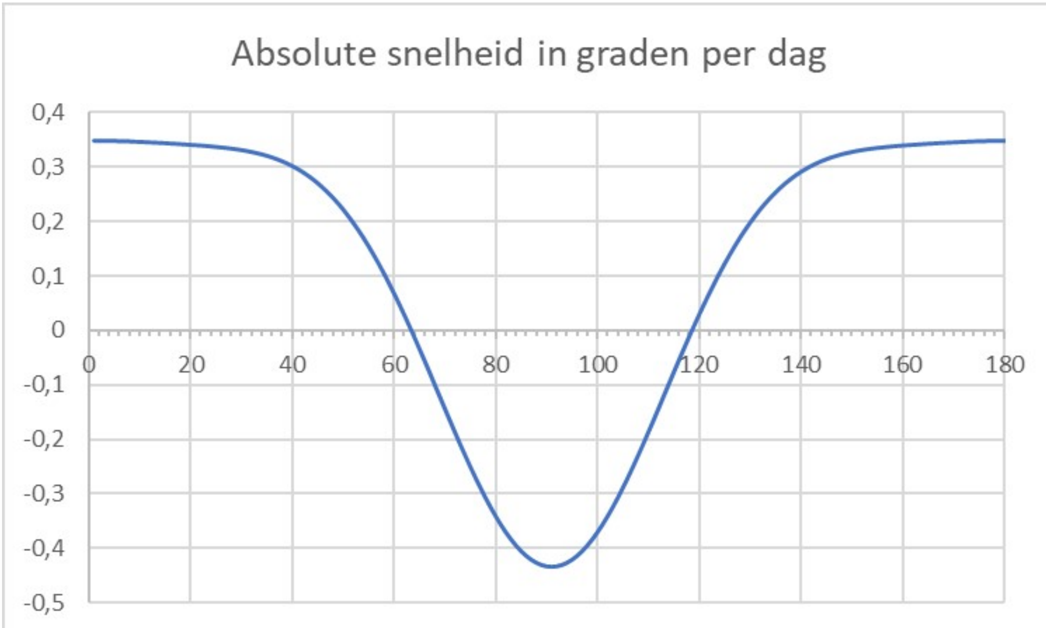

Figure 7: Velocitiy of the corrected Black Moon as a function of the angle between the Sun and the Black Moon

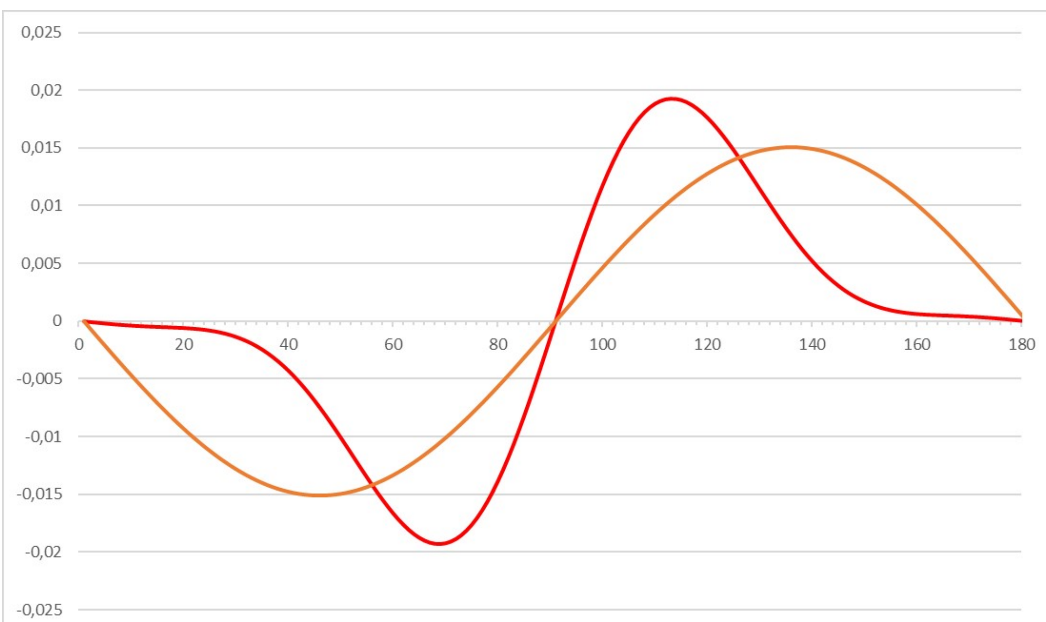

Figure 8: Acceleration of the corrected Black Moon (red curve) versus sine model (orange curve)

determine the acceleration of the cBM. Note the strongly time varying behaviour of the acceleration. The acceleration graph casts a glance at the interaction of gravitational forces of the Sun, the earth and the Moon. The phases are clearly distinguished:

$0° - 30°$	Small and slightly increasing deceleration.
$45° - 68°$	Strongly increasing deceleration.
$68° - 90°$	An almost twice as strong decrease of deceleration, that
$90° - 112°$	continues as an equally strong increase of acceleration.
$112° - 135°$	Strongly decreasing acceleration.
$150° - 180°$	Small and slightly decreasing acceleration.

The relevance of the acceleration lies in the connection with Newton's second law, that, popularly stated, equates the force acting on a body with the product of its mass and acceleration (for bodies with constant mass). With this connection, one of the components of the forces is revealed that Duval and Font must have used in their model of the cBM's position to compose their ephemerides.

4.3 An example

According to the ephemerides for 11^{th} July 2021 at 0.00 hrs, BM is at $29°\,9'$ in Taurus, and cBM is at $11°\,4'$ in Gemini. One day later, BM is at $29°\,16'$ in Taurus, and cBM is at $11°\,16'$ in Gemini. The position of the Sun is given by the computer program PlanetDance [5]. On these days at the same time, the Sun is in Cancer at $18°\,51'$ and $19°\,48'$ respectively. The angle between the Sun and the BM is $49°\,42'$ on the 11^{th}, and $50°\,32'$ one day later. With an angle of 50° between the Sun and the BM, and a daily motion of the Sun of $0°\,57'$, the calculations using (5) give a direct motion of the cBM of 12.34 arcminutes per day. From the ephemerides, that list values in integer degrees and arcminutes only, a direct motion of 12 arcminutes is obtained.

The cBM velocity is decreasing with a calculated deceleration of 35.6 arcseconds per day-squared on July 12^{th}. The deceleration grows to one arcminute per day-squared on July 26^{th}. The ephemerides list the beginning of cBM retrograde motion during the 26^{th} July 2021. A change of direction of motion is calculated to occur at an angle of $62° - 63°$ between the Sun and the BM, which takes place between between the 26^{th} and the 27^{th} of July.

5 Conclusions

A relatively simple formula for the longitude position in the zodiac of the corrected Black Moon is described in this paper, quite accurately producing values of Duval and Font's ephemerides. Therefore, the formula is very well suited for applications in Astrology. The mathematical approach, using simple sine functions is supported by a tool from signal analysis, using the Discrete Fourier Transform. Having obtained a mathematical expression, it was an easy exercise to determine the velocity and acceleration of this enigmatic celestial point. Knowledge of position, velocity and acceleration as functions of time does, however, not reveal the astronomical meaning is of the corrected Black Moon of Bode and Duval and Font's 'Lilith Vraie'. Certainly, it is not the lunar apogee, as can be deduced from modern ephemerides such as the Swiss Ephemerides [6]. Hence, the conclusion must be drawn that with the contribution of this paper the underlying model of Duval and Font's ephemerides is one step closer to disclosure, but still not fully understood.

References

[1] George Bode: Zwarte Maan Zwarte Zon Drakenkop, Stichting Vulcanus, Amsterdam, 2001, in Dutch.

[2] Max Duval & Jean-Marc Font: Tables du Nœud Lunaire de Lilith et du Soleil Noir, Dervy-Livres, Paris, 1988, in French.

[3] Harry Uitermark, personal communication.

[4] Wikipedia: Fourier Analysis, `https://en.wikipedia.org/wiki/Fourier_analysis`

[5] Jean Cremers: PlanetDance, `https://jcremers.com`

[6] The Swiss Ephemeris, `https://www.astro.com/swisseph/swephinfo_e.htm`

Appendix

Derivation of the Equations of Motion of the Corrected Black Moon

The lattitude correction angle of Equation (3) is written slightly more generally as follows.

$$\varphi_{cD} = a^\circ \sin\left[p\left(\varphi - b^\circ \sin(p\,\varphi)\right)\right] + c^\circ \sin(3p\,\varphi) \tag{7}$$

The constants $a^\circ = 12.37^\circ$, $b^\circ = 11.726^\circ$, $c^\circ = (8.8/60)^\circ$, are deliberately written with the degree symbol to indicate that the outcome of the equation is an angle in degrees. The use of degrees necessitates the application of a conversion factor $p = \frac{\pi}{90}$ for degrees to radians for the trigonometric functions of 2φ.

Equation (7) can be written in a more compact form as $F(\varphi) = f(\varphi) + e(\varphi)$. The first derivative of F is then calculated as follows.

$$\begin{aligned}
F'(\varphi) &= \frac{dF(\varphi)}{d\varphi} \\
&= a^\circ \cos\left[p\left(\varphi - b^\circ \sin(p\,\varphi)\right)\right] \cdot \left[p - p^2 b^\circ \cos(p\,\varphi)\right] + 3p\,c^\circ \cos(3p\,\varphi) \\
&= \underbrace{p\,a^\circ \cos\left[p\left(\varphi - b^\circ \sin(p\,\varphi)\right)\right]}_{g(\varphi)} \cdot \underbrace{\left[1 - p\,b^\circ \cos(p\,\varphi)\right]}_{h(\varphi)} + \underbrace{3p\,c^\circ \cos(3p\,\varphi)}_{e'(\varphi)} \qquad (8)
\end{aligned}$$

From (8) it is seen that the first derivative F' of F is dimensionless. We continue with $F'(\varphi) = g(\varphi)h(\varphi) + e'(\varphi)$ to determine the second derivative F'' of F.

$$F''(\varphi) = \frac{dF'(\varphi)}{d\varphi} = g'(\varphi)h(\varphi) + g(\varphi)h'(\varphi) + e''(\varphi) \tag{9}$$

The derivatives of the partial functions in (9) are:

$$\begin{aligned}
g'(\varphi) &= -p\,a^\circ \sin\left[p\left(\varphi - b^\circ \sin(p\,\varphi)\right)\right] \cdot \left[p - p^2 b^\circ \cos(p\,\varphi)\right] \\
&= -p^2 a^\circ \sin\left[p\left(\varphi - b^\circ \sin(p\,\varphi)\right)\right] \cdot h(\varphi) \\
&= -p^2 f(\varphi) \cdot h(\varphi) \qquad (10) \\
h'(\varphi) &= p^2 b^\circ \sin(p\,\varphi) \qquad (11) \\
e''(\varphi) &= -9p^2 c^\circ \sin(3p\,\varphi) \\
&= -9p^2 e(\varphi) \qquad (12)
\end{aligned}$$

Substitution of (10),(11), and (12) in (9) gives another compact expression for F''.

$$F''(\varphi) = -p^2 \left[f(\varphi)h^2(\varphi) + g(\varphi)b^\circ \sin(p\,\varphi) - 9e(\varphi)\right]. \tag{13}$$

From (13) it is seen that the dimension of F'' is $(\text{degree})^{-1}$, which complies with the expression for acceleration.